YEAR 12 Mathematics Ext 1
HSC Course
Assessment Task 1
December 2009

1. There are 4 sections.
2. Answer each question on your own paper showing all necessary working
3. Start each section on a new page
4. Calculators may be used

Topic	Mark
1. Section 1 (Parametric Equations)	/15
2. Section 2 (Mathematical Induction)	/11
3. Section 3 (Trigonometry)	/11
4. Section 4 (Harder Trigonometry)	/7

TOTAL /44

ASSESSMENT TASK 1 **YEAR 12 MATHEMATICS EXT 1** **DECEMBER 2009**

Start a new page for each section.

Section 1. Parametric Equations (15 marks)

i. Find the Cartesian equation of the following parabolas

a) $x = t$ (1 mark)

$y = \frac{t^2}{2}$

b) $x = t - 4$ (2 marks)

$y = 2 + 2t + t^2$

ii. Write down the parametric equations (in terms of t) for the parabola $x^2 - 8y = 0$ (2 marks)

iii. P$(4p, 2p^2)$ and Q$(4q, 2q^2)$ are two points on a parabola.

a) Show that the gradient of the chord PQ is $\frac{p+q}{2}$ (2 marks)

b) Hence show the equation of chord PQ is $y = \frac{1}{2}(p+q)x - 2pq$ (2 marks)

c) Show that if PQ is a focal chord then $pq = -1$ (1 mark)

iv. The point T$(16t, 8t^2)$ lies on the parabola $x^2 = by$

a) Find the value of b (1 mark)

b) Show the equation of the tangent at T is given by $y = tx - 8t^2$ (2 marks)

c) The tangent at T cuts the y axis at $(0, -128)$
What are the co-ordinates of T? (2 marks)

Section 2. Mathematical Induction (11 marks)

i. Prove by Mathematical induction

a) $9^n - 3$ is divisible by 6 for all positive integers n (5 marks)

b) $2^n > 2n + 1$, $n \geq 3$ (6 marks)

Section 3. Trigonometry (11 marks)

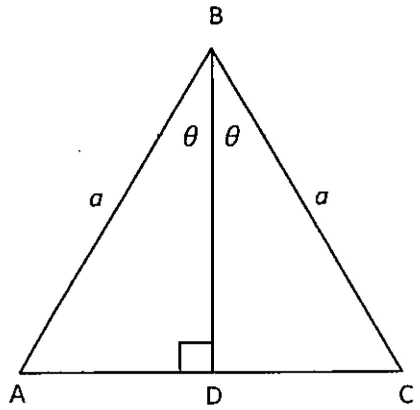
i. Given $f(x) = \sin^3 x + cos^3 x$ (2 marks)
Show that when $f(x) = 0$, $tanx = -1$

ii. Write $cos\theta - \sqrt{3}sin\theta$ in the form $R\cos(\theta + \alpha)$ where α is in degrees (5 marks)
Hence solve $cos\theta - \sqrt{3}sin\theta = 1 \;\; for \;\; 0° \leq \theta \leq 360°$

iii. Solve by the t method , where $\boldsymbol{t = tan\frac{\theta}{2}}$
$3sin\theta - 4cos\theta = 4 \;\; for \;\; 0° \leq \theta \leq 360°$ (4 marks)
(Give solutions to θ to the nearest degree)

Section 4. Harder Trigonometry (7 marks)

i. The triangle ABC is isosceles with AB = BC = a, and BD is perpendicular to AC.
Let $\angle ABD = \angle CBD = \theta$



a) By considering the **area** of ΔABD **only**, show that its area is $\dfrac{a^2 sin\theta cos\theta}{2}$ (2 marks)
b) By considering the **area of** ΔABC , prove that $sin2\theta = 2sin\theta cos\theta$ (2 marks)

ii. Prove that $\dfrac{sin5\theta}{sin\theta} - \dfrac{cos5\theta}{cos\theta} = 4cos2\theta$ (3 marks)

END OF EXAMINATION

Q1/

(i)(a) $y = \frac{x^2}{2}$

$x^2 = 2y$

(b) $y = 2 + 2(x+4) + (x+4)^2$ $\quad x = t - 4$

$x + 4 = t$

$y = 2 + 2x + 8 + x^2 + 8x + 16$

$y = x^2 + 10x + 26$

(ii) $x^2 = 8y$ $\quad a = 2$

using $(2at, at^2)$

$(4t, 2t^2)$

$x = 4t$ $\quad y = 2t^2$

(iii) $\frac{2p^2 - 2q^2}{4p - 4q} = \frac{2(p-q)(p+q)}{4(p-q)}$

$= \frac{p+q}{2}$

(c) $y - 2p^2 = \frac{p+q}{2}(x - 4p)$

$y - 2p^2 = \frac{1}{2}(p+q)x - 2p^2 - 2pq$

$y = \frac{1}{2}(p+q)x - 2pq$

(d) Sub $(0, 2)$ $\quad 2 = -2pq$

$-1 = pq$ (as required)

(iv) (a) $256t^2 = 8bt^2$

$8b = 256$

$b = 32$

(b) $x^2 = 32y$

$y = \frac{x^2}{32}$

$\frac{dy}{dx} = \frac{2x}{32}$

$= \frac{x}{16}$ at $x = 16t$

$m_T = t$

$y - 8t^2 = t(x - 16t)$

$y = tx - 16t^2 + 8t^2$

$y = tx - 8t^2$

(c) $-128 = 0 - 8t^2$

$t^2 = 16$

$t = \pm 4$ $\therefore$ T is $(64, 128)$ $(-\ldots$

Question 2

Step 1 Prove true for $n=1$

ie $9^1 - 3 = 6$ (1)

$\therefore$ true for $n=1$

Step 2 Assume true for $n=k$ where $k \in J$

ie $9^k - 3 = 6M$ where $M \in J$

$9^k = 6M + 3$ ($1\frac{1}{2}$)

Step 3 Prove true for $n = k+1$

ie $9^{k+1} - 3$ is divisible by 6

$$\begin{aligned} 9^{k+1} - 3 &= 9.9^k - 3 \\ &= 9(6M+3) - 3 \\ &= 54M + 27 - 3 \\ &= 54M + 24 \\ &= 6(9M+4) \end{aligned}$$

$\therefore$ true for $n = k+1$ ($2\frac{1}{2}$)

Step 4 As it is true for $n=1$, and if true for $n=k$, it is true for $n=k+1$.
$\therefore$ true for all positive integers of n.

(b) Step 1 Prove true for $n=3$

$2^3 > 2(3)+1$

$8 > 7$

$\therefore$ true for $n=3$

Step 2 Assume true for $n=k$, where k is a positive integer greater than or equal to 3

ie $2^k > 2k+1$

Step 3 Prove true for $n=k+1$

ie $2^{k+1} > 2(k+1)+1$

$2^{k+1} > 2k+3$

$2^k . 2 > 2k+3$

$(2k+1).2 > 2k+3$

from assumption

$4k+2 > 2k+3$

As $k \geq 3$ the smallest difference that can occur for $4k - 2k$ is 6.

$\therefore 4k+2 > 2k+3$

$\therefore$ true for $n = k+1$

Step 4 As true for $n=3$ and if true for $n=k$, it is true for $n=k+1$ $\therefore$ true for all positive integers for $n \geq 3$

Question 3

(i) $\frac{\sin^3 x}{\cos^3 x} + \frac{\cos^3 x}{\cos^3 x} = \frac{0}{\cos^3 x}$

$\tan^3 x + 1 = 0$

$\tan^3 x = -1$

$\therefore \tan x = -1$

(ii) $\cos\theta - \sqrt{3}\sin\theta$

$R = \sqrt{3+1}$
$= 2$

$\tan\alpha = \sqrt{3}$
$\alpha = 60°$

$= 2\cos(\theta + 60°)$

$2\cos(\theta+60) = 1$

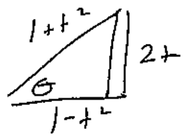
$\cos(\theta + 60°) = \frac{1}{2}$

$60° \le \theta + 60 \le \cancel{360}\ 420°$

$\theta + 60° = 60°, 300°, 420°$

$\theta = 0°, 240°, 360°$

(iii) $\frac{3(2t)}{1+t^2} - 4\frac{(1-t^2)}{1+t^2} = 4$



$6t - 4 + 4t^2 = 4 + 4t^2$

$6t = 8$

$t = \frac{4}{3}$

$\tan\frac{\theta}{2} = \frac{4}{3}$

$\frac{\theta}{2} = 53°, \cancel{23}$

$\theta = 106°$

Check $\theta = 180°$

$3\sin 180 - 4\cos 180$

$= 0 - -4$

$= 4$

$\therefore$ Sol^s are $106°$ and $180°$

(iv) $\frac{\sin 5\theta}{\sin\theta} - \frac{\cos 5\theta}{\cos\theta} = 4\cos 2\theta$

$LHS = \frac{\sin 5\theta\cos\theta - \cos 5\theta \sin\theta}{\sin\theta\cos\theta}$

$= \frac{\sin(5\theta - \theta)}{\sin\theta\cos\theta}$

$= \frac{\sin 4\theta}{\sin\theta\cos\theta}$

$= \frac{2\sin 2\theta\cos 2\theta}{\frac{1}{2}\sin 2\theta}$

$= 4\cos 2\theta$

$= RHS$

(iv) (a) Area $\Delta ABD = \frac{1}{2} . AD . BD$

$\sin\theta = \frac{AD}{a}$ $\qquad$ $\cos\theta = \frac{BD}{a}$

$a\sin\theta = AD$ $\qquad$ $a\cos\theta = BD$

$\therefore \quad A = \frac{1}{2} \, a\sin\theta \, . \, a\cos\theta$

$= \frac{a^2 \sin\theta \cos\theta}{2}$

(b) Area $\Delta ABC = 2 \times$ Area ΔABD

$= a^2 \sin\theta \cos\theta$

also Area $\Delta ABC = \frac{1}{2} \times a \times a \times \sin 2\theta$

$= \frac{1}{2} a^2 \sin 2\theta$

$\therefore$

$\frac{1}{2} a^2 \sin 2\theta = a^2 \sin\theta \cos\theta$

$\frac{1}{2} \sin 2\theta = \sin\theta \cos\theta$

$\sin 2\theta = 2 \sin\theta \cos\theta$